ANALYSIS OF HYPERSONIC GAS FLOW PAST BLUNT BODIES

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An approximate method of investigation of a flow past very blunt bodies, the face of which little differs from their transverse cross-section, is given. Assumptions of Newton's theory and solutions in which stream tubes of varying cross-section [1] appear, are not valid for the class of bodies under consideration. Unlike the numerical methods of integration of differential equations given in [2], our method is iterative, and we present approximate formulas expressing the pressure and separation of the shock wave from the body, as a function of degree of compression.

1. We shall consider a plane or axially symmetric problem of homogeneous hypersonic flow of an ideal gas (see Fig.1). Here ABC is the contour of the body, 1 is the inner region adjacent to the surface of the body and 2 is the outer region adjacent to the shock wave DOK. Let u°, v°, p° and ρ° be the longitudinal and transverse components of the velocity vector, pressure and density, respectively, in the region between the shock wave and the body.

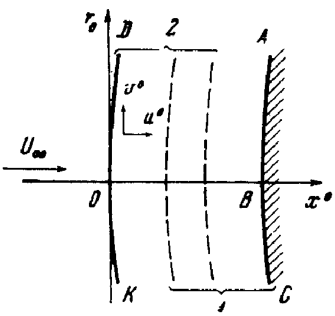

Fig. 1

Flow of gas is described by Euler equations, by the continuity equations on the shock wave and by the boundary condition of nonturbulency of the flow. Parameters of the unperturbed flow shall be denoted by ∞. Solution of our problem can be obtained by integrating two systems of differential equations separately. One of these systems is valid in the region 1, the other in the region 2. Since mathematical relationship between these solutions exist, they can be combined to describe the complete flow field.

Let us introduce the parameter

$$\varepsilon = \frac{\gamma - 1}{\gamma + 1} + \frac{2}{(\gamma + 1) M_\infty^2} \qquad \left(\gamma = \frac{c_p}{c_v}\right)$$

where γ is the ratio of specific heats.

The condition that the equations of motion of the gas and relations on the shock wave form a nontrivial system when $\varepsilon \to 0$, leads to the following transformation of coordinates:

$$x^\circ = \varepsilon^{1/2} x, \qquad r^\circ = r \tag{1.1}$$

and of the velocity components, pressure and density according to respective formulas

$$\frac{u^\circ}{U_\infty} = \varepsilon u\,(x,\ r) + O\,(\varepsilon^2), \qquad \frac{v^\circ}{U_\infty} = \varepsilon^{1/2} v\,(x,\ r) + O\,(\varepsilon^{3/2})$$
$$\frac{p^\circ - p_\infty}{\rho_\infty U_\infty^2} = 1 + \varepsilon p\,(x,\ r) + O\,(\varepsilon^2), \quad \left(\frac{\rho_\infty}{\rho^\circ}\right)^{-1} = \frac{1}{\varepsilon} + \sigma\,(x,\ r) + O\,(\varepsilon) \tag{1.2}$$

The limit $\varepsilon \to 0$ is taken when $\gamma \to 1$ and $M_\infty \to \infty$.
Substitution of (1.1) and (1.2) into Euler's equations results in the following system of differential equations for the outer region:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} = -\frac{\partial p}{\partial x}, \quad u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial r} = 0$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial r} = -\frac{jv}{r^j}, \quad \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial r}\right)(p - \sigma) = 0 \tag{1.3}$$

Here $j = 0$ for the plane, and $j = 1$ for axially symmetric flows. Boundary conditions on the shock wave (index c) are written thus

$$v_c = \frac{dx_c}{dr_c}, \qquad u_c = 1 + v_c^2 = -p_c \tag{1.4}$$

It can be shown that $(v,\ r)$ are characteristic coordinates. Then, equations of the system (1.4) can be reduced to canonical form

$$u_v^*\,(u^* - vx_r^*) + vx_v^*u_r^* = -p_v^*, \quad u^* - vx_r^* = 0, \quad u_v^* - x_r^* = -\frac{jv}{r^j}x_v^* \tag{1.5}$$

which can easily be transformed into

$$vx_v^*u_r^* = -p_v^*, \quad \frac{\partial}{\partial v}(r^jx_r^* + jx^*) = 0, \quad u^* = vx_r^* \tag{1.6}$$

Second equation of (1.3) can be integrated

$$x^* = \frac{1}{r^j}f'\,(v) + g\,(r) \tag{1.7}$$

Consequently

$$u^* = v\left[g'\,(r) - \frac{j}{r^{j+1}}f'\,(v)\right]$$
$$p^* = G\,(r) - \frac{g''\,(r)}{r^j}\int_{v_b}^{v} t^2 f''\,(t)\,dt - \frac{j\,(j+1)}{r^{j+2}}\int_{v_b}^{v} t^2 f''\,(t)\,f'\,(t)\,dt \tag{1.8}$$

where $f(v)$ and $g(r)$ are arbitrary functions of integration, and $G(r)$ is an arbitrary function, as will be shown later, of pressure distribution on the body. Lower limit in the integrals (1.8) is the value of velocity v on the body (index b) and will be equal to zero in case of a nonturbulent flow. In this manner we have obtained a complete solution of the given problem, in terms of the functions f, g and G.

Equation of the shock wave in parametric form is

$$r_c = r_c\,(v), \qquad x_c^* = x^*\,[v,\ r_c\,(v)] \tag{1.9}$$

Then, first relation of (1.4) can be rewritten thus

$$v\,\frac{dr_c}{dv} = \frac{dx_c^*}{dv} \tag{1.10}$$

Substitution $x^* = x_c^*$ into (1.7), gives

$$\left(v - \frac{dg}{dr_c} + \frac{j}{r_c^{j+1}}\right)\frac{dr_c}{dv} = \frac{1}{r_c^j}f''\,(v) \tag{1.11}$$

Combining (1.4) and (1.8) we obtain an equation connecting unknown functions $g(r)$ and $f(v)$ on the surface of the shock wave

$$g'\,(r_c) - \frac{j}{r_c^{j+1}}f'\,(v) = v + \frac{1}{v} \tag{1.12}$$

This enables us to write (1.11) as

$$r_c^j \frac{dr_c}{dv} = -vf''(v) \tag{1.13}$$

which is integrable, and gives

$$f(v) - vf'(v) = -v^2 \frac{d}{dv}\left(\frac{f}{v^2}\right) = \frac{r_c^{j+1}}{j+1} \tag{1.14}$$

2. Using the relations (1.8) to (1.14) we first obtain a particular solution. The equation connecting $f(v)$ and $g(r)$ on the shock wave, is very complex. Near the stagnation point however, the values of r and v are small. Consequently, unknown functions can be represented by series, used below to connect the inner and outer solution. Let the plane case be characterized by the x and y variables, while the axisymmetric case – by x and r. In this section we shall only consider the first case. By symmetry, the equation of the shock wave $y_c(v)$ can be represented by a series in odd powers of v

$$y_c(v) = Av + Bv^3 + O(v^5) \tag{2.1}$$

Let us substitute this equation into the right-hand side of (1.14). Resulting differential equation can, for $j = 0$, be easily integrated with respect to $f(v)$

$$f(v) = k_1 v - Av \ln v - {}^1/_2 Bv^3 + O(v^5) \qquad (v \to 0) \tag{2.2}$$

Apart from this, from (1.12) we obtain

$$g(y_c) = A \ln y_c + a_1 + \frac{A+B}{A^2}\frac{y_c^2}{2} + O(y_c^3) \quad (y_c \to 0) \tag{2.3}$$

and analogously

$$x_c^* = k_1 - A + a_1 + A \ln A + {}^1/_2 Av^2 + O(v^4) \tag{2.4}$$

Suitable choice of the origin of coordinates, gives

$$k_1 - A + a_1 + A \ln A = 0 \tag{2.5}$$

Consequently, the parametric representation of the shock wave will be

$$y_c(v) = Av + Bv^3 + O(v^5), \qquad x_c^*(v) = {}^1/_2 Av^2 + O(v^4) \tag{2.6}$$

Finally, the solution is given in terms of asymptotic series

$$x^*(v, y) = A \ln \frac{y}{Av} + \frac{A+B}{A^2}\frac{y^2}{2} - \frac{3}{2}Bv^2 + O(v^4) \qquad (v, y \to 0)$$

$$u^*(v, y) = v\left(\frac{A}{y} + \frac{A+B}{A^2}y\right) + O(v^3) \qquad (y \to 0) \tag{2.7}$$

$$p^*(v, y) = G(y) - \left(\frac{A}{y^2} - \frac{A+B}{A^2}\right)\left(\frac{Av^2}{2} + \frac{3}{4}Bv^4\right) + \ldots$$

Pressure on the shock wave has a relatively simple form

$$p_c^*(v, y_c) = G(y_c) - \left[{}^1/_2 - \left(1 + \frac{3}{2}\frac{B}{A}\right)\frac{v^2}{2}\right] \tag{2.8}$$

From the second relation of (1.4) we see, that when $v \to 0$,

$$G(0) = -{}^1/_2 \tag{2.9}$$

It can easily be checked that the solutions (2.7) and (2.8) differ from the corresponding results of [2].

3. Let us now extend the results obtained in [2] to the axially symmetric case ($j = 1$). Approximations and few other details will, in this case, differ somewhat from those in the two-dimensional case. Instead of the first equation of (2.6), we shall have

$$r_c = Av + Bv^3 + O(v^5) \tag{3.1}$$

and correspondingly

$$f(v) = -{}^1/_2 A^2v^2 - {}^1/_3 ABv^4 + O(v^6), \quad f'(v) = -A^2v - {}^4/_3 ABv^3 + O(v^5) \tag{3.2}$$

From (1.12) we obtain

$$g(r_c) = A + \frac{1}{2A}\left(1 + \frac{2}{3}\frac{B}{A}\right) r_c^2 + O(r_c^4) \qquad (3.3)$$

and analogously

$$x_c^* = {}^1/_2 A v^2 + {}^3/_4 B v^4 + O(v^6) \qquad (3.4)$$

Arbitrary constant is chosen here in such a manner, that when $r = 0$, $x_c^* = 0$. Equation of the shock wave in parametric form will be

$$r_c(v) = Av + Bv^3 + O(v^5), \qquad x_c^*(v) = {}^1/_2 A v^2 + O(v^4) \qquad (3.5)$$

The solution can be expressed in terms of the following expansions, valid in the vicinity of stagnation point:

$$x^*(v, r) = A - A^2\frac{v}{r} + \frac{1}{2A}\left(1 + \frac{2}{3}\frac{B}{A}\right) r^2 - \frac{4}{3}AB\frac{v^3}{r} + \ldots \qquad (3.6)$$

$$u^*(v, r) = v\left[A^2\frac{v}{r}\left(1 + \frac{4}{3}\frac{B}{A}v^2\right) + \frac{1}{A}\left(1 + \frac{2}{3}\frac{B}{A}\right) r + \frac{B}{A^4}\left(\frac{B}{A} - 1\right) r^3\right] + \ldots$$

$$p^*(v, r) = G(r) + \frac{A}{3}\left(1 + \frac{2}{3}\frac{B}{A}\right)\frac{v^3}{r} - \frac{16}{9}A^3B\frac{v^6}{r^4} - \frac{A^4v^4}{2r^4} + \ldots$$

The last of these expansions gives on the shock wave

$$p_c^* = G(r_c) - \frac{1}{2} + \left(1 + \frac{4}{3}\frac{B}{A}\right)\frac{v^2}{3} + O(v^4) \qquad (G(0) = -{}^1/_2) \qquad (3.7)$$

In conclusion we shall comment on the arbitrariness in obtaining the functions $f'(v)$ from (1.11). It can easily be shown that an arbitrary constant k_1 can be added to $f'(v)$ and $k_1 r^j$ substituted from $g(r)$ without any loss of generality. In fact, this method was used in the derivation of asymptotic relationships for f and g in (3.2) and (3.3).

4. We shall now proceed to construct a solution for the inner region, which will be valid near the surface of the body. Let the contour of the body ($j = 0$) be described by the function $x_b(y)$. In the plane case, the system of differential equations coincides with that [2] obtained for the flow past a flat plate at right angles to the direction of the flow. In our case however, the solution of differential equations will be different, due to different boundary conditions, and will be of the form

$$V = V_b(y)\cosh\lambda(X - X_b) \qquad (4.1)$$

$$U = V_b(y)X_b'(y)\cosh\lambda(X - X_b) - \frac{1}{\lambda}V_b'(y)\sinh\lambda(X - X_b) \qquad (4.2)$$

It will satisfy a simple boundary condition on the body when $X = X_b(y)$

$$\frac{dX_b}{dy} = \frac{U}{V} \qquad (4.3)$$

Pressure can be found from Bernoulli's integral

$$P_b(y) = -{}^1/_2 - {}^1/_2 V^2_b(y) \qquad (4.4)$$

As in previous formulas, subscript b denotes the values of the body, and

$$X = \frac{x^\circ - D(\varepsilon)}{\mu(\varepsilon)}$$

where $D(\varepsilon)$ is the distance of separation of the shock wave from the body, $\mu(\varepsilon)$ is a quantity of the same order as the width of the inner region. Transverse velocity found from the inner solution of (4.1) when $X \to -\infty$ will now agree with the value obtained from the outer solution (2.7) and (2.4) when $x^* \to \infty$. Utilizing the basic condition for combining the solutions ($\lambda = 1/A$), we obtain

$$g(y) = A\ln\varepsilon^{1/2}\frac{V_b(y)}{2} + X_b(y) + \text{const} \qquad (4.5)$$

This equation plays the part of the inner boundary condition for the outer solution, since the velocity distribution $V_b(y)$ is related to the outer solution by the condition of pressure compatibility, i.e. it requires that (1.8) should coincide with (4.4) for the values $j = 0$ and $r = y$ In this case we can easily confirm that

$$V_b(y) = \sqrt{3(1 + B/2A)}\, V \qquad (4.6)$$

Putting (4.6) into (4.5) and taking (2.4) into account, we obtain

$$D(\varepsilon) = \frac{1}{2} \varepsilon^{1/2} A \ln \frac{4}{3\varepsilon(1 + {}^1/_2 B/A)} \qquad (4.7)$$

The condition of equality of pressure which was obtained above for a limited region, should be extended to the body in order to obtain a complete solution for the plane case. This can be done by equating $V_b(y)$ which defines the pressure on the body, to the transverse velocity v on the shock wave using the same values of y, and (4.5). Let us consider a function $v(y_0)$, being the inverse of $y_0(v)$. In what follows we shall use the symbol v without the subscript o, to denote velocity on the shock wave for some definite values of y. Differentiating (4.5) and putting this into (1.12), yields

$$g'(y) = v + \frac{1}{v} = A\frac{V_b'(y)}{V_b(y)} + X_b'(y) \qquad (4.8)$$

When $j = 0$, (1.12) gives

$$g''(y) = -\frac{1 - v^2}{v^2}\frac{dv}{dy} \qquad (4.9)$$

Next we shall compute the magnitude $[p_c - G(y)] / g''$. To do this, we shall utilize (1.4) together with (1.8), (4.4) and (4.8). Assuming $G(y)$ to be equal to $P_b(y)$, we obtain

$$\frac{P_c - P_b}{g''(y)} = \frac{{}^1/_2(1 + 2v^2 - V_b^2)}{(1 - v^2)\, v^{-2} dv/dy} = -\int_0^v t^2 f''(t)\, dt \qquad (4.10)$$

Replacing $tf''(t)$ under the integral sign with dv/dy given by (1.13) and differentiating (4.9) with respect to v, we finally obtain

$$v = \frac{1}{2}\frac{d}{dy}\left\{\frac{v^2(1 + 2v^2 - V_b^2)}{(1 - v^2)\, dv/dy}\right\} \qquad (4.11)$$

which together with (4.7), forms a system of two equations for $v(y)$ and $v_b(y)$ Determination of these functions solves the problem of obtaining the shock wave equation and velocity distribution for the given body.

5. We shall now investigate the inner solution for the axially symmetric case. It can easily be shown that differential equations remain the same as in the plane case, i.e. [2], with the exception of continuity equation

$$\partial U / \partial X + \partial V / \partial r + V / r = 0 \qquad (5.1)$$

Let $x_b(r)$ be the equation of the generator of the solid of revolution. We shall assume the inner solution to remain valid only in the region, in which $X - g(r)$ or $X(r) - X_b(r)$ are of the order of $\varepsilon^{1/2}$ together with $U - gV$ or $U - X_b V$. Besides $\partial V/\partial r$ can also be of the $O(\varepsilon^{1/2})$, if $X_b \neq 0$. With the above restrictions, we can use the coordinate system and transformed variables of the case $j = 0$. It is easy to see that the system of differential equations for the inner region has a solution

$$V = V_b(r) - \lambda_* r(X - X'_b)$$
$$U = V_b X_b(r) - (X - X_b)(V_b' + V_b / r + \lambda_* r X_b') + \lambda_* (X - X_b)^2 \qquad (5.2)$$

satisfying the boundary condition $U = X_b'(r)V$ on the surface of the body. Pressure is given by

$$P_b(r) = -{}^1/_2 - {}^1/_2 V_b^2 \qquad (5.3)$$

Combining transverse velocities in a suitable manner leads to the requirement, that $\varepsilon^{1/2} V(r)$ from (5.2) should agree with c $v \sim rA^{-2}(g - x^*)$, obtained from the second relation of (3.2) and (1.7). This makes it possible to find λ_*, which is

$$\lambda_* = \frac{1}{A^2}\varepsilon^{-1/2}, \qquad g(r) = X_b(r) + \varepsilon^{1/2}A^2\frac{V_b(r)}{r} \tag{5.4}$$

We shall now return to the particular solution, to obtain the expression for the separation of the shock wave from the body. (3.7), (3.5) and (1.4) yield

$$V_b(r) = \left[\frac{8}{3}\left(1 + \frac{1}{3}\cdot\frac{B^*}{A}\right)\right]^{1/2}\frac{r}{A} + O(r^3) \tag{5.5}$$

Hence, from (5.4) and (3.3) we finally obtain

$$D(\varepsilon) = A\varepsilon^{1/2}\left\{1 - \left[\frac{8\varepsilon}{3}\left(1 + \frac{B}{3A}\right)\right]^{1/2}\right\} \tag{5.6}$$

Complete solution of the problem can be obtained, similarly to the case of $j = 0$, by setting up a number of differential equations. In the axially symmetric case two integrals in the expression (1.8) for the pressure are removed by differentiating (1.8) twice. This leads to high order of the resulting system and complexity, sufficient to require numerical methods of integration. The fact that the inner layer and displacement thickness $g(r) - X_b(r)$ are both small, cannot be utilized. A better and simpler method follows: outer solution is obtained by integrating

$$\left(1 - \frac{1}{v^2}\right) r_c \frac{dv}{dr_c} + \frac{1}{v} + 2v = 2\frac{dx_b(r_c)}{dr_c} + r_c\frac{d^2x_b(r_c)}{dr_c^{\,2}} \tag{5.7}$$

with boundary condition $x_b(r) = g(r)$ which can be obtained using (1.12), (1.11) and (1.7) to find x^*, and condition on the shock wave for the transverse velocity

$$v_c = \left(\frac{dx^*}{dr}\right)_c$$

Equation (5.7) was investigated in [1], therefore we can assume that the first approximation is known. Pressure on the body can be expressed by Formula

$$p_b^* = p_c^*(r) - \frac{x_b^*(r)}{r}\int_0^{x_c^*}\xi dx_c^*(\xi) + \frac{2}{r^3}\int_0^{x_c^*}\xi\left[x_b^*(r) - x^*(r, x_c^*(\xi))\right]dx_c^*(\xi) \tag{5.8}$$

$$x^*(r, x_c^*) = x^*(r, V) \qquad \text{for} \qquad x_c^* = x_c^*(V)$$

First integral in the formula for pressure is proportional to the Newtonian impulse and gives a centrifugal term dependent on the curvature of the body. Before integrating the following term, we must equate $(x_b^* - x^*)/r^2$ to the curvature of the stream line relative to the surface of the body. Then, the second term can be considered as a centrifugal term resulting from the above curvature. In other words, function $G(r)$ is computed by extracting definite integrals in (1.13) and suitable adjustment of pressure along the shock wave. The value of $V_b(r)$ is found next from (5.3), and the iterative process continues with a new value of $g(r)$ found from (5.4). This leads to more accurate equation of the shock wave and better value of transverse velocity $V_b(r)$ on the body.

6. As an example, we shall solve the problem of flow past a plane disk with $X_b'(r) = 0$. We assume zero approximation to be known from [1]. Then

$$g(r) = A = 3^{3/4} r_* \tag{6.1}$$

where the subscript asterisk $*$ denotes the critical value of r.

Besides,

$$r_c = \frac{Av}{(1 + 2v^2)^{3/4}} \tag{6.2}$$

This is sufficient to give, together with (1.10)

$$x_c^* = A\left[1 - \frac{1+v^2}{(1+2v^2)^{3/4}}\right] \tag{6.3}$$

According to (1.11), we have

$$f'(v) = -A^2 \frac{v(1+v^2)}{(1+2v^2)^{3/2}} \tag{6.4}$$

The value of the integral in (1.13) is easily found

$$\int_0^{\tau} t^2 f''(t) f'(t)\,dt = \frac{A^4}{32(1+2v^2)^3}[2v^2(1+9v^2+10v^4)-(1+2v^2)^3\ln(1+2v^2)] \tag{6.5}$$

Using (4.4) we can now find v_b^2, and obtain the negative value of the thickness $g(r) - \chi(r)$ from (5.4). On the axis, $B/A = -{}^3/_2$, hence, taking (5.6) into account, we obtain

$$D(\varepsilon) = A\varepsilon^{1/2}\left(1 - \sqrt{{}^4/_3\varepsilon}\right) \tag{6.6}$$

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